

TORSION CLASSES GENERATED BY SILTING MODULES

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ABSTRACT. We study the classes of modules which are generated by a silting module. In the case of either hereditary or perfect rings it is proved that these are exactly the torsion \mathcal{T} such that the regular module has a special \mathcal{T} -preenvelope. In particular every torsion enveloping class in $\text{Mod-}R$ are of the form $\text{Gen}(T)$ for a minimal silting module T . For the dual case we obtain for general rings that the covering torsion-free classes of modules are exactly the classes of the form $\text{Cogen}(T)$, where T is a cosilting module.

1. INTRODUCTION

Silting modules were introduced in [3] in order to extend the τ -tilting theory, developed in [1] and [11] (for finitely generated modules over artin algebras), to infinitely generated modules. The dual notion, i.e. cosilting modules, was studied in [8]. A module $T \in \text{Mod-}R$ is a (co)silting module if the class of all T -(co)generated modules is a torsion (torsion-free) class of a special form.

It was proved in [1, Proposition 1.1 and Section 2] that in the case of finitely generated modules over artin algebras the classes of the form $\text{gen}(T)$ (i.e. epimorphic images of finite direct sums of copies of T) induced by a τ -tilting module coincide to the torsion classes which are enveloping. We refer to [6, Section 5] for similar characterizations in the (co)tilting cases. As in the case of infinitely generated (co)tilting modules, [5], a natural question is to ask for characterizations of torsion classes which are of the form $\text{Gen}(T)$ (respectively $\text{Cogen}(T)$) for a (co)silting module T . Here $\text{Gen}(T)$ ($\text{Cogen}(T)$) denotes the the closure to isomorphisms of the class of all quotients (submodules) of direct sums (products) of copies of T . We recall that silting modules are in bijection with silting objects in the derived category of $\text{Mod-}R$ which can be represented by complexes of the form $0 \rightarrow P_{-1} \rightarrow P_0 \rightarrow 0$ with P_{-1} and P_0 projectives. Therefore, they are also in correspondence with important concepts as (co-)t-structures or simply-minded collections of objects (see [12] and [3]). It was proved recently that for some classes of rings (e.g. hereditary or commutative ring), they can be parametrized by universal localisations, [13], Gabriel topologies of finite type, [2], or wide subcategories of finitely presented modules [4]. For other correspondences and constructions we refer to [14] and [15]. For various correspondences in the cosilting case, we refer to [18] and [19].

In this note we provide a general characterization (Proposition 2.1) for silting classes, as torsion classes which are generated via some special pushout constructions. In the case when R is right perfect (Theorem 2.4), respectively right hereditary (Theorem 2.6) it leads to characterizations which can be viewed as extensions of the corresponding result for tilting classes, [5, Theorem 2.1]. In particular, every

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enveloping torsion class of modules over a perfect ring or over an hereditary ring is generated by a silting module (Corollary 2.9). The case of perfect rings extends the corresponding results proved for finitely generated modules over artin algebra in [16] and [1, Theorem 2.7].

The last section of the paper is devoted to the dual setting, namely, we consider torsion-free classes which are of the form $\text{Cogen}(T)$, where T is a cosilting module. Since injective modules form an enveloping class over a general ring, we obtain using dual tools that for every ring R torsion-free covering classes in $\text{Mod-}R$ are exactly the classes which are cogenerated by cosilting modules (Theorem 3.5).

2. SILTING CLASSES

In this paper R is a unital ring, and $\text{Mod-}R$ will denote the category of all right R -modules. Moreover, \mathcal{P} will be the class of all projective modules in $\text{Mod-}R$ and \mathcal{P}^\rightarrow will denote the class of all homomorphisms $\sigma : P_{-1} \rightarrow P_0$ with $P_{-1}, P_0 \in \mathcal{P}$.

If $\sigma : P_{-1} \rightarrow P_0$ is a homomorphism from \mathcal{P}^\rightarrow then we can associate to σ the class

$$\mathcal{D}_\sigma = \{X \in \text{Mod-}R \mid \text{Hom}_R(\sigma, X) \text{ is an epimorphism}\}.$$

If T is a right R -module then $\text{Gen}(T)$ denotes the class of all epimorphic images of direct sums of copies of T .

If \mathcal{T} is a class of modules, we will use the following classes:

- ${}^\circ\mathcal{T} = \{X \in \text{Mod-}R \mid \text{Hom}(X, T) = 0 \text{ for all } T \in \mathcal{T}\},$
- $\mathcal{T}^\circ = \{X \in \text{Mod-}R \mid \text{Hom}(T, X) = 0 \text{ for all } T \in \mathcal{T}\},$
- ${}^\perp\mathcal{T} = \{X \in \text{Mod-}R \mid \text{Ext}^1(X, T) = 0 \text{ for all } T \in \mathcal{T}\},$
- $\mathcal{T}^\perp = \{X \in \text{Mod-}R \mid \text{Ext}^1(T, X) = 0 \text{ for all } T \in \mathcal{T}\},$
- $\square\mathcal{T} = \{\alpha \in \mathcal{P}^\rightarrow \mid \mathcal{T} \subseteq \mathcal{D}_\alpha\},$ and
- ${}^\diamond\mathcal{T} = \{\text{Coker}(\alpha) \mid \alpha \in \square\mathcal{T}\}.$

Recall from [3] that a module T is *presilting* if there exists a projective presentation

$$P_{-1} \xrightarrow{\sigma} P_0 \rightarrow T \rightarrow 0$$

such that \mathcal{D}_σ is a torsion class and $T \in \mathcal{D}_\sigma$. Then $\text{Gen}(T) \subseteq \mathcal{D}_\sigma \subseteq T^\perp$ and $(\text{Gen}(T), T^\circ)$ is a torsion pair. If $\mathcal{D}_\sigma = \text{Gen}(T)$ then T is called a *silting module*.

Let \mathcal{T} be a class of modules. Then a homomorphism $\epsilon : X \rightarrow T$ with $T \in \mathcal{T}$ is a \mathcal{T} -preenvelope if $\text{Hom}(\epsilon, T')$ is surjective for all $T' \in \mathcal{T}$, i.e. all homomorphisms $X \rightarrow T'$ with $T' \in \mathcal{T}$ factorize through ϵ . The \mathcal{T} -preenvelope ϵ is a \mathcal{T} -envelope if every homomorphism $\alpha : T \rightarrow T$ with the property $\epsilon = \alpha\epsilon$ has to be an isomorphism. If all modules $X \in \text{Mod-}R$ have a \mathcal{T} -preenvelope (envelope) then \mathcal{T} is *preenveloping* (resp. *enveloping*). A preenveloping class \mathcal{T} is *special* if for every $X \in \text{Mod-}R$ we can find a \mathcal{T} -preenvelope ϵ which is monic and $\text{Coker}(\epsilon) \in {}^\perp\mathcal{T}$. The corresponding dual notions are that of (special) precover/precovering and cover/covering, respectively.

Tilting classes, i.e. the torsion classes of the form $\text{Gen}(T)$ with T a tilting module, can be characterized by the fact that they are exactly the special preenveloping torsion classes in $\text{Mod-}R$ (cf. [6, Section 5], [5, Theorem 2.1]). We refer to [9] for a recent study of this kind of special preenveloping situation which involves homomorphisms instead of objects. Even the orthogonality used in this paper does not cover the (co)silting case (cf. [7, Remark 39]), we can characterize torsion

classes of the form $\text{Gen}(T)$ with T a silting module by the existence of preenvelopes with some special properties.

Proposition 2.1. *The following are equivalent for a class \mathcal{T} of R -modules:*

- (1) *There exists a silting module T such that $\mathcal{T} = \text{Gen}(T)$;*
- (2) (a) *\mathcal{T} is a torsion class,*
 (b) *there exists a \mathcal{T} -preenvelope $\epsilon : R \rightarrow M$ which can be obtained as a pushout*

$$\begin{array}{ccccccc} L_{-1} & \xrightarrow{\rho} & L_0 & \longrightarrow & K & \longrightarrow & 0 \\ \downarrow \lrcorner & & \downarrow & & \parallel & & \\ R & \xrightarrow{\epsilon} & M & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

such that $\rho \in \square \mathcal{T}$.

- (3) (a) *\mathcal{T} is a torsion class,*
 (b) *for every R -module X there exists an \mathcal{T} -preenvelope $\epsilon' : X \rightarrow T'_0$ which can be obtained as a pushout*

$$\begin{array}{ccccccc} L_{-1} & \xrightarrow{\rho} & L_0 & \longrightarrow & T_1 & \longrightarrow & 0 \\ \downarrow \lrcorner & & \downarrow & & \parallel & & \\ X & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & 0 \end{array}$$

such that $\rho \in \square \mathcal{T}$.

If we have a diagram as in (2) then $M \oplus K$ is a silting module and $\mathcal{T} = \text{Gen}(T) = \mathcal{D}_\rho$.

Proof. (1) \Rightarrow (3) Let $\sigma : P_{-1} \rightarrow P_0$ be a homomorphism from \mathcal{P}^\rightarrow such that $T = \text{Coker}(\sigma)$, and T is silting with respect to σ . Hence $\mathcal{T} = \mathcal{D}_\sigma$.

For every module X we consider the canonical homomorphism $\delta : P_{-1}^{(I)} \rightarrow X$, where $I = \text{Hom}_R(P_{-1}, X)$, and we construct the pushout diagram

$$\begin{array}{ccccccc} P_{-1}^{(I)} & \xrightarrow{\sigma^{(I)}} & P_0^{(I)} & \longrightarrow & T^{(I)} & \longrightarrow & 0 \\ \downarrow \lrcorner & & \downarrow \delta_0 & & \parallel & & \\ X & \xrightarrow{\mu} & T_0 & \xrightarrow{\nu} & T^{(I)} & \longrightarrow & 0 \end{array}$$

Then, as in the proof of [3, Theorem 3.12] we obtain that $T_0 \in \mathcal{D}_\sigma = \mathcal{T}$.

Moreover, for every $Y \in \mathcal{T}$ and every homomorphism $\alpha : X \rightarrow Y$ there exists $\beta : P_0^{(I)} \rightarrow Y$ such that $\delta\alpha = \beta\sigma^{(I)}$. By the pushout universal property there exists $\gamma : T_0 \rightarrow Y$ such that $\alpha = \gamma\mu$, hence μ is a \mathcal{T} -preenveloping map. Since $\sigma^{(I)} \in \square \mathcal{T}$, the proof is complete.

(3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1) If $X \in \mathcal{D}_\rho$ then every homomorphism $R \rightarrow X$ can be lifted to a homomorphism $M \rightarrow X$. Therefore every element of X is in the image of a homomorphism $M \rightarrow X$, hence $X \in \text{Gen}(M)$. It follows that $\mathcal{D}_\rho \subseteq \text{Gen}(M) = \mathcal{T}$. But $\mathcal{T} \subseteq \mathcal{D}_\rho$ since $\rho \in \square \mathcal{T}$, and it follows that $\mathcal{D}_\rho = \mathcal{T}$ is a torsion class. Moreover, $K \in \mathcal{T} = \mathcal{D}_\rho$, hence K is presilting with respect to ρ . By the proof of [3, Theorem 3.12] it follows that $T = M \oplus K$ is a silting module T such that $\text{Gen}(T) = \mathcal{D}_\rho = \mathcal{T}$. \square

In order to apply the above proposition we will use the following characterization for pushout diagrams.

Lemma 2.2. *In a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\iota} & L_{-1} & \xrightarrow{\rho} & L_0 & \longrightarrow & T_1 & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \delta & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & U & \xrightarrow{v} & X & \xrightarrow{\xi} & T_0 & \longrightarrow & T_1 & \longrightarrow & 0 \end{array}$$

the middle square is a pushout if and only if α is an epimorphism.

Proof. Suppose that the middle square is a pushout. If α is not an epimorphism, we consider $\pi : U \rightarrow U/\text{Im}(\alpha)$ the canonical epimorphism, and $\mu : U/\text{Im}(\alpha) \rightarrow E$ is the embedding of $U/\text{Im}(\alpha)$ into its injective envelope. There exists a homomorphism $\nu : X \rightarrow E$ such that $\nu v = \mu\pi$, hence $\nu\delta\iota = 0$. Then νv factorizes through ρ . Since the middle square is a pushout, ν factorizes through ξ . It follows that $\mu\pi = \nu v = 0$. Since μ is monic, we obtain $\pi = 0$, hence α is an epimorphism.

Conversely, if α is an epimorphism and we have two homomorphisms $\beta_1 : X \rightarrow Y$ and $\beta_2 : L_0 \rightarrow Y$ such that $\beta_1\delta = \beta_2\rho$ then $\beta_1v\alpha = 0$, hence $\beta_1v = 0$. It follows that there exists a unique homomorphism $\bar{\beta} : \text{Im}(\xi) \rightarrow Y$ such that $\beta_1 = \bar{\beta}v$.

Let $\bar{\delta} : \text{Im}(\rho) \rightarrow \text{Im}(\xi)$ be the homomorphism induced by δ . If $\iota_\rho : \text{Im}(\rho) \rightarrow L_0$ and $\iota_\xi : \text{Im}(\xi) \rightarrow T_0$ are the canonical inclusions, then $\bar{\beta}\bar{\delta} = \beta_2\iota_\rho$. Since the first square in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(\rho) & \xrightarrow{\iota_\rho} & L_0 & \longrightarrow & T_1 \longrightarrow 0 \\ & & \downarrow \bar{\delta} & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Im}(\xi) & \xrightarrow{\iota_\xi} & T_0 & \longrightarrow & T_1 \longrightarrow 0 \end{array}$$

is a pushout, there exists a unique homomorphism $\beta^* : T_0 \rightarrow Y$ such that $\bar{\beta}$ (hence β_1) and β_2 factorize through β^* , and the proof is complete. \square

Let Y be a submodule of a module P , and consider a canonical projection $\pi : P \rightarrow P/Y$. Recall that $Y \ll P$ means that Y is a *superfluous* submodule of a module P , i.e. that for every $\varphi \in \text{Hom}(M, P)$ if $\pi\varphi$ is an epimorphism then φ is an epimorphism.

Make a well-known easy observation:

Lemma 2.3. *Let X, P, T be modules over a ring R such that $X \ll P$ and $\alpha \in \text{Hom}(P, T)$. Then $\alpha(X) \ll \alpha(P)$. If, furthermore, $\alpha(X) = \alpha(P)$ then $\alpha = 0$.*

Now we are ready to characterize torsion classes generated by silting modules over right perfect rings.

Theorem 2.4. *Let R be a right perfect ring and $\mathcal{T} \subseteq \text{Mod-}R$ a torsion class. The following are equivalent:*

- (1) $\mathcal{T} = \text{Gen}(T)$ for a silting module T ;
- (2) There exists a \mathcal{T} -preenvelope $\epsilon : R \rightarrow M$ such that $M \in \mathcal{T} \cap {}^\perp \mathcal{T}$.

In these conditions, if $K = \text{Coker}(\epsilon)$ then $M \oplus K$ is a silting module, and $\mathcal{T} = \text{Gen}(M \oplus K)$.

Proof. (1) \Rightarrow (2) This is a consequence of Proposition 2.1 (see also [3, Proposition 3.11]).

(2) \Rightarrow (1) We consider the exact sequence $0 \rightarrow U \rightarrow R \xrightarrow{\epsilon} M \xrightarrow{\rho} K \rightarrow 0$. If $\bar{\epsilon} : R/U \rightarrow M$ is the homomorphism induced by ϵ then for every $T \in \mathcal{T}$ the homomorphism $\text{Hom}(\bar{\epsilon}, T)$ is an epimorphism. Since $M \in {}^\perp \mathcal{T}$ we obtain $K \in {}^\perp \mathcal{T}$.

For an epimorphism $\gamma : P_0 \rightarrow M$ with P_0 projective, we have a commutative diagram

$$\begin{array}{ccccccc} P_{-1} & \xrightarrow{\sigma} & P_0 & \longrightarrow & K & \longrightarrow & 0 \\ \downarrow \pi_\sigma & & \parallel & & \parallel & & \\ 0 \longrightarrow & Z & \xrightarrow{\bar{\sigma}} & P_0 & \longrightarrow & K & \longrightarrow 0 \\ \downarrow \bar{\delta} & & \downarrow \gamma & & \parallel & & \\ 0 \longrightarrow & R/U & \xrightarrow{\bar{\epsilon}} & M & \longrightarrow & K & \longrightarrow 0, \end{array}$$

where $\bar{\sigma}$ and $\bar{\epsilon}$ are the canonical homomorphisms induced by σ and ϵ , respectively, and $\pi_\sigma : P_{-1} \rightarrow Z$ is the projective cover of Z .

Since P_{-1} is projective we can construct

$$\begin{array}{ccccccc} 0 \longrightarrow & X & \xrightarrow{\iota_X} & P_{-1} & \xrightarrow{\sigma} & P_0 & \longrightarrow K \longrightarrow 0 \\ \downarrow v & & \downarrow \delta & & \downarrow \gamma & & \parallel \\ 0 \longrightarrow & U & \xrightarrow{\iota_U} & R & \xrightarrow{\epsilon} & M & \longrightarrow K \longrightarrow 0 \end{array}$$

such that $\pi_U \delta = \bar{\delta} \pi_\sigma$, where $\pi_U : R \rightarrow R/U$ is the canonical projection. The maps ι_X and ι_U are the inclusion maps. Moreover, $X \ll P_{-1}$.

We will show that $\text{Hom}(\sigma, T)$ is onto for all $T \in \mathcal{T}$. Let us consider the induced short exact sequence

$$0 \rightarrow Z \rightarrow P_0 \rightarrow K \rightarrow 0,$$

where $Z = \text{Im}(\sigma) \cong P_{-1}/X$, and note that for every $T \in \mathcal{T}$ we have a short exact sequence

$$(*) \quad 0 \rightarrow \text{Hom}(K, T) \rightarrow \text{Hom}(P_0, T) \xrightarrow{\text{Hom}(\sigma, T)} \text{Hom}(Z, T) \rightarrow 0$$

since $\text{Ext}^1(K, T) = 0$. Fix an arbitrary $T \in \mathcal{T}$ and an arbitrary $\varphi \in \text{Hom}(P_{-1}, T)$. Let us denote by $\pi_X : P_{-1} \rightarrow P_{-1}/X$ and $\pi_T : T \rightarrow T/\varphi(X)$ the canonical projections. Then we can find a homomorphism $\bar{\varphi} \in \text{Hom}(P_{-1}/X, T/\varphi(X))$ which satisfies $\bar{\varphi} \pi_X = \pi_T \varphi$. As $T/\varphi(X) \in \mathcal{T}$, there exists $\bar{\psi} \in \text{Hom}(P_0, T/\varphi(X))$ for which $\bar{\psi} \sigma = \bar{\varphi} \pi_X$ by the exactness of $(*)$. Since P_0 is projective and π_T is an epimorphism, $\bar{\psi}$ factorizes through π_T , i.e. there exists $\psi \in \text{Hom}(P_0, T)$ such that $\pi_T \psi = \bar{\psi}$. Hence

$$\pi_T \psi \sigma = \bar{\psi} \sigma = \bar{\varphi} \pi_X = \pi_T \varphi.$$

Put $\alpha := \varphi - \psi \sigma \in \text{Hom}(P_{-1}, T)$. From $\pi_T \alpha = 0$ we have $\alpha(P_{-1}) \subseteq \varphi(X)$. Furthermore, $\alpha|_X = \varphi|_X$ since $\psi \sigma(X) = 0$, which implies that $\alpha(P_{-1}) \subseteq \alpha(X)$. By Lemma 2.3 we obtain $\alpha = 0$, so $T \in \mathcal{D}_\sigma$.

Using the pushout of σ and δ we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & P_{-1} & \xrightarrow{\sigma} & P_0 & \longrightarrow & K & \longrightarrow & 0 \\
& & \downarrow v' & & \downarrow \delta & \lrcorner & \downarrow \gamma' & & \parallel & & \\
0 & \longrightarrow & V & \longrightarrow & R & \xrightarrow{\epsilon'} & L & \xrightarrow{\rho'} & K & \longrightarrow & 0 \\
& & \downarrow \bar{v} & & \parallel & & \downarrow \bar{\gamma} & & \parallel & & \\
0 & \longrightarrow & U & \longrightarrow & R & \xrightarrow{\epsilon} & M & \xrightarrow{\rho} & K & \longrightarrow & 0,
\end{array}$$

such that $\bar{v}v' = v$ and $\gamma = \bar{\gamma}\gamma'$. In order to simplify the presentation, let us remark that v' is surjective and \bar{v} is injective, hence V can be identified to the image of v . In this case the equality $\bar{v}v' = v$ represents the canonical decomposition of v through its image.

We will prove that $\bar{\gamma}$ is a \mathcal{T} -preenvelope for L . Similar techniques were also used in [17]. For every $T \in \mathcal{T}$ and every homomorphism $\alpha : L \rightarrow T$ there exists $\bar{\alpha} : M \rightarrow T$ such that $\alpha\epsilon' = \bar{\alpha}\epsilon$. Then $(\bar{\alpha}\bar{\gamma} - \alpha)\epsilon' = 0$, hence there exists $\beta : K \rightarrow T$ such that $\bar{\alpha}\bar{\gamma} - \alpha = \beta\rho' = \beta\rho\bar{\gamma}$. Then $\alpha = (\bar{\alpha} - \beta\rho)\bar{\gamma}$, and the claim is proved. Moreover, since $M \in {}^\perp\mathcal{T}$, if we use the exact sequence $0 \rightarrow \text{Ker}(\bar{\gamma}) \rightarrow L \rightarrow M \rightarrow 0$ it follows that $\text{Hom}(\text{Ker}(\bar{\gamma}), T) = 0$ for all $T \in \mathcal{T}$.

We split the bottom rectangle in the previous commutative diagram in two commutative diagrams with short exact sequences,

$$\begin{array}{ccccccc}
0 & \longrightarrow & V & \longrightarrow & R & \xrightarrow{\epsilon'} & \text{Im}(\epsilon') \longrightarrow 0 \\
& & \downarrow \bar{v} & & \parallel & & \downarrow \zeta \\
0 & \longrightarrow & U & \longrightarrow & R & \xrightarrow{\epsilon} & \text{Im}(\epsilon) \longrightarrow 0,
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Im}(\epsilon') & \longrightarrow & L & \xrightarrow{\rho'} & K \longrightarrow 0 \\
& & \downarrow \zeta & & \downarrow \bar{\gamma} & & \parallel \\
0 & \longrightarrow & \text{Im}(\epsilon) & \longrightarrow & M & \xrightarrow{\rho} & K \longrightarrow 0,
\end{array}$$

where ζ can be identified to the canonical surjection $R/V \rightarrow R/U$. Applying Ker-Coker Lemma, we observe that $C = \text{Coker}(\bar{v}) \cong \text{Ker}(\bar{\gamma})$.

If $\pi : P \rightarrow C$ is a projective cover for C and $\alpha : P \rightarrow T$ is a homomorphism with $T \in \mathcal{T}$ then the induced homomorphism

$$\bar{\alpha} : P/\text{Ker}(\pi) \rightarrow T/\alpha(\text{Ker}(\pi)), \quad \bar{\alpha}(x + \text{Ker}(\pi)) = \alpha(x) + \alpha(\text{Ker}(\pi)),$$

is zero. Therefore $\alpha(P) = \alpha(\text{Ker}(\pi))$. Since $\text{Ker}(\pi)$ is superfluous, it follows that $\alpha = 0$, hence $\text{Hom}(P, \mathcal{T}) = 0$.

We lift π to a homomorphism $\bar{\pi} : P \rightarrow U$. By Lemma 2.2, v' is an epimorphism, and it is easy to see that $\text{Im}(v) + \text{Im}(\bar{\pi}) = \text{Im}(\bar{v}) + \text{Im}(\bar{\pi}) = U$. Therefore, the canonical map $(\bar{\pi}, v) : P \oplus X \rightarrow U$ induced by $\bar{\pi}$ and v is an epimorphism.

Now we construct the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P \oplus X & \xrightarrow{1_P \oplus \iota_X} & P \oplus P_{-1} & \xrightarrow{(0, \sigma)} & P_0 & \longrightarrow & K \longrightarrow 0 \\
& & \downarrow (\bar{\pi}, \nu) & & \downarrow (\iota_U \bar{\pi}, \delta) & & \downarrow \gamma & & \parallel \\
0 & \longrightarrow & U & \xrightarrow{\iota_U} & R & \xrightarrow{\epsilon} & M & \longrightarrow & K \longrightarrow 0.
\end{array}$$

Since $(0, \sigma) \in {}^\square \mathcal{T}$, it remains to apply Lemma 2.2 and Proposition 2.1 to complete the proof. \square

The following class of examples, used also in [2], shows that the implication (2) \Rightarrow (1) does not hold in general.

Example 2.5. Let R be a local ring with the unique maximal ideal J such that $J^2 = J \neq 0$ and put $S = R/J$. Since idempotency of J implies that extensions of semisimple modules by semisimple modules are semisimple over R as well, we get that $\mathcal{T} = \text{Gen}(S) = \{S^{(\kappa)} \mid \kappa\}$ is a torsion class and $\text{Ext}^1(T, U) = 0$ for each $T, U \in \mathcal{T}$. It is easy to see that the natural projection $R \rightarrow S$ forms a \mathcal{T} -envelope of R . Clearly $S \in \mathcal{T} \cap {}^\perp \mathcal{T}$ and we will show that $S^{(\kappa)}$ is not silting for every cardinal $\kappa > 0$.

Consider an exact sequence $P_{-1} \xrightarrow{\sigma} P_0 \xrightarrow{\rho} S^{(\kappa)} \rightarrow 0$. Since ρ factorizes through the canonical projection $\pi : R^{(\kappa)} \rightarrow S^{(\kappa)}$ we may suppose that $P_0 = R^{(\kappa)}$ and $\rho = \pi$. Then $P_{-1} \neq 0$ and $\text{Im}(\sigma) = J^{(\kappa)} = P_0 J$. Hence for every $T \in \mathcal{T}$ and every homomorphism $\varphi \in \text{Hom}(P_0, T)$ we have $\text{Im}(\sigma) = P_0 J \subseteq \ker(\varphi)$, and so $\varphi \sigma = 0$. As $\text{Hom}(\sigma, T) = 0$ while $\text{Hom}(P_{-1}, T) \neq 0$ for all nonzero $T \in \mathcal{T}$, we obtain that no generator of \mathcal{T} is silting.

In the case of hereditary rings, silting torsion classes can be characterized by the existence of a special long exact sequence.

Theorem 2.6. *Let R be a right hereditary ring and $\mathcal{T} \subseteq \text{Mod-}R$ a torsion class. The following are equivalent:*

- (1) $\mathcal{T} = \text{Gen}(T)$ for a silting module T ;
- (2) There exists a \mathcal{T} -preenvelope $\epsilon : R \rightarrow M$ such that $M \in \mathcal{T} \cap {}^\perp \mathcal{T}$.
- (3) There exists an exact sequence $0 \rightarrow U \rightarrow R \rightarrow M \rightarrow K \rightarrow 0$ such that $M \in \mathcal{T}$, $U \in {}^\circ \mathcal{T}$ and $K \in {}^\perp \mathcal{T}$.

Proof. (2) \Rightarrow (3) As in the proof of Theorem 2.4 we obtain $K \in {}^\perp \mathcal{T}$.

Since ϵ is a \mathcal{T} -preenvelope, every homomorphism $R \rightarrow T$ with $T \in \mathcal{T}$ factorizes through R/U . Therefore, for every $T \in \mathcal{T}$ we have that $\text{Hom}(\pi, T)$ is an isomorphism, where $\pi : R \rightarrow R/U$ is the canonical epimorphism. Then first natural homomorphism from the exact sequence

$$0 \rightarrow \text{Hom}(R/U, T) \rightarrow \text{Hom}(R, T) \rightarrow \text{Hom}(U, T) \rightarrow \text{Ext}^1(R/U, T)$$

is an isomorphism. Moreover, using the exact sequence $0 \rightarrow R/U \rightarrow M \rightarrow K \rightarrow 0$, we obtain $\text{Ext}^1(R/U, T) = 0$ for all $T \in \mathcal{T}$. Therefore $\text{Hom}(U, T) = 0$ for all $T \in \mathcal{T}$.

(3) \Rightarrow (1) Since R is hereditary, there exists a projective resolution

$$0 \rightarrow P_{-1} \xrightarrow{\sigma} P_0 \rightarrow K \rightarrow 0.$$

Using the hypothesis $K \in {}^\perp \mathcal{T}$, it follows that $\sigma \in {}^\square \mathcal{T}$. If $U = \text{Ker}(\epsilon)$ we can construct, as in the proof of Theorem 2.4, the projectivity of P_0 , a commutative

diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & U \oplus P_{-1} & \xrightarrow{\sigma} & P_0 & \longrightarrow & K & \longrightarrow & 0 \\
& & \parallel & & \downarrow (\iota, \delta) & & \downarrow \gamma & & \parallel & & \\
0 & \longrightarrow & U & \longrightarrow & R & \xrightarrow{\epsilon} & M & \longrightarrow & K & \longrightarrow & 0,
\end{array}$$

where $\iota : U \rightarrow R$ is the inclusion map. Since U is projective, by $U \in {}^\circ\mathcal{T}$ it follows that $(0, \sigma) \in {}^\square\mathcal{T}$. From Proposition 2.1 we conclude that $\text{Gen}(T) = \mathcal{T}$ is a silting class. \square

The class of silting modules is an intermediate class between the class of tilting modules and that of quasi-tilting modules. Using a theorem of Wei, [20], it is proved in [3, Proposition 3.15] that in the case of finitely generated modules over finitely dimensional algebras the silting finitely generated modules coincide to (finendo) quasi-tilting modules. In the case of hereditary or right perfect rings we obtain a similar result for the classes generated by silting, respectively finendo quasi-tilting modules.

Corollary 2.7. *Let R be a right hereditary or right perfect ring. The following are equivalent for a torsion class $\mathcal{T} \subseteq \text{Mod-}R$:*

- (1) $\mathcal{T} = \text{Gen}(T)$ for a silting module T ;
- (2) $\mathcal{T} = \text{Gen}(T)$ for a finendo quasi-tilting module T .

Proof. This follows from the previous theorems by using [3, Proposition 3.10] and [3, Theorem 3.2(3)]. \square

We recall from [4] that a silting module T is *minimal* if there exists a $\text{Gen}(T)$ -envelope for the regular module R . In order to apply the above results to minimal silting modules we need a lemma whose proof is a simple exercise.

Lemma 2.8. *Let \mathcal{T} a class of modules. If $\epsilon : R \rightarrow M$ is an \mathcal{T} -envelope then every epimorphism $\alpha : N \rightarrow M$ with $N \in \mathcal{T}$ splits.*

Consequently, if \mathcal{T} is a class closed under extensions and $\epsilon : R \rightarrow M$ is an \mathcal{T} -envelope then $M \in {}^\perp\mathcal{T}$.

Corollary 2.9. *The following are equivalent for a torsion class \mathcal{T} of modules over a right hereditary or right perfect ring R :*

- (1) $\mathcal{T} = \text{Gen}(T)$ for a minimal silting module T ;
- (2) There exists a \mathcal{T} -envelope $\epsilon : R \rightarrow M$.

In particular, all enveloping torsion classes over hereditary or right perfect rings are generated by silting modules.

Moreover, a half of Salce's Lemma [10, Lemma 5.20] is valid for silting modules:

Proposition 2.10. *Let T be a silting module. If $\mathcal{T} = \text{Gen}(T)$ then for every R -module X there exists a short exact sequence*

$$0 \rightarrow L \rightarrow U \xrightarrow{v} X \rightarrow 0$$

such that v is a ${}^\circ\mathcal{T}$ -precover for X and $L \in \mathcal{T}$.

Consequently, ${}^\perp\mathcal{T}$ is a special precovering class.

Proof. If X is an R -module, we consider a pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{v} & P & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \parallel \\ 0 & \longrightarrow & L & \xrightarrow{\gamma} & U & \longrightarrow & X \longrightarrow 0, \end{array}$$

where P is a projective module and α is a \mathcal{T} -preenvelope for Y obtained as a pushout

$$\begin{array}{ccccccc} P_{-1} & \xrightarrow{\zeta} & P_0 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow v' & & \downarrow & & \parallel \\ Y & \xrightarrow{\alpha} & L & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

for some $\zeta \in {}^\square\mathcal{T}$. Then we have a pushout square

$$\begin{array}{ccc} P_{-1} & \xrightarrow{vv'} & P \\ \downarrow \zeta & & \downarrow \beta \\ P_0 & \xrightarrow{\gamma\gamma'} & U, \end{array}$$

hence U is the cokernel of the homomorphism $\delta : P_{-1} \rightarrow P_0 \oplus P$ induced by vv' and ζ . Since every homomorphism $f : P_{-1} \rightarrow T$ with $T \in \mathcal{T}$ can be written as $f = g\zeta$ for some $g : P_0 \rightarrow T$, it follows that $f = g'\delta$, where $g' : P_0 \oplus P \rightarrow T$ is defined by $g'_{|P_0} = g$ and $g'_{|P} = 0$. Then $\delta \in {}^\square\mathcal{T}$, so $U \in {}^\diamond\mathcal{T}$.

Now, for every $V \in {}^\diamond\mathcal{T}$ we have $\mathcal{T} \subseteq V^\perp$, and it follows that γ is a ${}^\diamond\mathcal{T}$ -precover for X .

The last statement follows from the inclusion ${}^\diamond\mathcal{T} \subseteq {}^\perp\mathcal{T}$. \square

3. COSILTING CLASSES

For the dual results, let us recall from [8] that we can associate to every homomorphism $\sigma : Q_0 \rightarrow Q_1$ between injective modules the class

$$\mathcal{B}_\sigma = \{X \in \text{Mod-}R \mid \text{Hom}_R(X, \sigma) \text{ is an epimorphism}\},$$

and a module T is *partial cosilting* if there exists an injective presentation

$$0 \rightarrow T \rightarrow Q_0 \xrightarrow{\sigma} Q_1$$

such that \mathcal{B}_σ is a torsion-free class and $T \in \mathcal{B}_\sigma$. Then $\text{Cogen}(T) \subseteq \mathcal{B}_\sigma \subseteq {}^\perp T$. If $\mathcal{B}_\sigma = \text{Cogen}(T)$ then T is called *cosilting*.

Let \mathcal{I} be the class of all injective modules, and \mathcal{I}^\rightarrow the class of all homomorphisms between injective modules. If \mathcal{F} is a class of right R -modules then we associate to \mathcal{F} the following classes

- $\mathcal{F}^\square = \{\sigma : S_0 \rightarrow S_1 \mid \sigma \in \mathcal{I}^\rightarrow, \text{ and } \mathcal{F} \subseteq \mathcal{B}_\sigma\}$, and
- $\mathcal{F}^\diamond = \{\text{Ker}(\sigma) \mid \sigma \in \mathcal{F}^\square\}$.

In order to dualize Theorem 2.4 and Corollary 2.9 let us formulate dual versions of Propositions 2.1.

Proposition 3.1. *The following are equivalent for a class \mathcal{F} of R -modules:*

- (1) *There exists a cosilting module T such that $\mathcal{F} = \text{Cogen}(T)$;*
- (2) (a) *\mathcal{F} is a torsion-free class,*

- (b) If E is a fixed injective cogenerator for $\text{Mod-}R$ then there exists an \mathcal{F} -precovering $\epsilon : M \rightarrow E$ which can be obtained as a pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{\epsilon} & E \\ & & \parallel & & \downarrow & & \downarrow \nu \\ 0 & \longrightarrow & K & \longrightarrow & Q'_0 & \xrightarrow{\zeta'} & Q'_1 \end{array}$$

such that $\zeta' \in \mathcal{F}^\square$.

- (3) (a) \mathcal{F} is a torsion-free class,
 (b) for every R -module X there exists an \mathcal{F} -precovering $\alpha : M \rightarrow X$ which can be obtained as a pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & S & \longrightarrow & M & \xrightarrow{\alpha} & X \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S & \longrightarrow & S_0 & \xrightarrow{\sigma} & S_1 \end{array}$$

such that $\sigma \in \mathcal{F}^\square$.

If we have a diagram as in (2) then $K \oplus M$ is a cosilting modules and $\mathcal{F} = \text{Cogen}(K \oplus M)$.

If Y is a submodule of a module P with the canonical embedding $\nu : Y \rightarrow P$, then Y is an essential submodule of P , $Y \trianglelefteq P$, if an arbitrary homomorphism $\varphi \in \text{Hom}(P, N)$ is a monomorphism whenever $\varphi\nu$ is a monomorphism.

Lemma 3.2. Let Y, Q, F be modules over a ring R such that $Y \trianglelefteq Q$ and $\alpha \in \text{Hom}(F, Q)$. Then $\beta(F) \cap Y \trianglelefteq \beta(F)$. If, furthermore, $\beta(F) \cap Y = 0$, then $\beta = 0$.

Lemma 3.3. If \mathcal{F} is a class closed under extensions and $\epsilon : M \rightarrow E$ is an \mathcal{F} -cover of an injective module E then $M \in \mathcal{F}^\perp$.

As in the (co)tilting theory, from [10, Theorem 5.31] and [8, Corollary 4.8] we obtain the following:

Lemma 3.4. If T is a cosilting module then $\text{Cogen}(T)$ is a covering class.

Since every module has an injective envelope over an arbitrary ring, application of dual techniques to that applied in the silting case gives us the dual result to Corollary 2.9 that cosilting classes are exactly that torsion-free classes which are covers over general rings.

Theorem 3.5. Let R be a ring and E a fixed injective cogenerator for $\text{Mod-}R$. If \mathcal{F} is a torsion-free class in $\text{Mod-}R$, the following are equivalent:

- (1) $\mathcal{F} = \text{Cogen}(T)$ for a cosilting module T ;
- (2) \mathcal{F} is a covering class;
- (3) there exists an \mathcal{F} -cover $\epsilon : M \rightarrow E$;
- (4) There exists an \mathcal{F} -precover $\epsilon : M \rightarrow E$ such that $M \in \mathcal{F} \cap \mathcal{F}^\perp$.

Moreover, if R is hereditary, then the above conditions are equivalent to:

- (5) There exists an exact sequence $0 \rightarrow K \rightarrow M \rightarrow E \rightarrow V \rightarrow 0$ such that $M \in \mathcal{F}$, $V \in \mathcal{F}^\circ$ and $K \in \mathcal{F}^\perp$.

In these conditions, if $K = \text{Ker}(\epsilon)$ then $M \oplus K$ is a cosilting module and $\mathcal{F} = \text{Cogen}(M \oplus K)$.

Proof. The implications of (1) \Rightarrow (2) follows from Lemma 3.4. The other implications in a cyclic proof are dual to those presented in the silting case. \square

We note that the equivalence (1) \Leftrightarrow (2) was proved independently by Zhang and Wei, cf. [18, Theorem 3.5] and [19, Theorem 4.18].

We have also the dual of Proposition 2.10.

Proposition 3.6. *Let T be a cosilting module. If $\mathcal{F} = \text{Cogen}(T)$ then for every R -module X there exists a short exact sequence*

$$0 \rightarrow X \xrightarrow{v} U \rightarrow F \rightarrow 0$$

such that v is a \mathcal{F}^\diamond -preenveloping for X and $F \in \mathcal{F}$.

Corollary 3.7. *Let $\mathcal{F} = \text{Cogen}(T)$ for a cosilting module T . Then the class*

$$\mathcal{F}^\perp = \{X \in \text{Mod-}R \mid \text{Ext}_R^1(F, X) = 0 \text{ for all } F \in \mathcal{F}\}$$

is an enveloping class.

Proof. Since $\mathcal{F}^\diamond \subseteq \mathcal{F}^\perp$, it follows that every \mathcal{F}^\diamond -preenvelope constructed in the previous proposition is a special \mathcal{F}^\perp -preenvelope. Therefore, it is enough to apply [10, Theorem 5.27] and [8, Corollary 4.8] to obtain the conclusion. \square

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